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Contact symmetries and integrable non-linear dynamical systems

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Abstract. We present a systematic method of classifying and constructing invariants for Lagrangians containing arbitrary polynomial non-linear potentials. It is based on the assumption that these Lagrangians are invariant under contact groups of transformations. For a finite number of degrees of freedom we can prove integrability for a large class of polynomial potentials. The method can be extended in several directions.

1. Introduction

One of the outstanding problems in theoretical physics nowadays lies in the study of non-linear dynamical systems and in particular the identification of those which are integrable. A fundamental question along this line is obviously to find a systematic way of finding which features are common to all those systems sharing this property. At the same time it would also be extremely interesting to develop a method to discern whether or not a dynamical system is integrable. Various alternatives have been proposed. Some authors (Ablowitz *et al* 1980, Ramani *et al* 1982) have pointed out that one of the common features of integrable systems is that their solutions have the Painlevé property. According to this conjecture a number of non-linear dynamical systems have been studied, some of them possessing the so-called naive Painlevé property and others obeying several refinements of this same property. Certainly Painlevé's conjecture must have something to do with integrability but it is not known, at least to the present authors, whether this conjecture can be finally established as a general property of these dynamical systems.

A different proposal (not neccessarily unrelated to the previous one) is that they admit some extra invariance group as a dynamical symmetry which could be a signal of integrability. However, point symmetries do not seem to be of any use here and we have to move to more general kinds of symmetry groups, among which the so-called contact groups appear to be an interesting choice. These symmetries contain coordinates and velocities in the transformation and have been studied in different frameworks for many years (Cerveró and Boya 1975, see also Campbell 1966). In a recent paper (Sahadevan and Lakshmanan 1986) the idea of using contact symmetry groups in non-linear dynamical systems with few degrees of freedom has been developed with success. These authors are able to recover two well known integrable systems (the Henon-Heiles potential and the two coupled anharmonic oscillators) and to find their first integrals as well. If such a relationship between contact symmetries and integrability can finally be established, an important step towards the complete classification of

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integrable non-linear dynamical systems would be possible, at least for a finite number of degrees of freedom. This paper is intended to be another contribution in this direction.

We study then, for some general potentials, those systems possessing a dynamical symmetry of the kind described above. Our conclusion is that for the potentials in the range studied, only those fulfilling the latter conjecture are integrable. Indeed, it would be nice to find a sort of proof that this property must be true for all possible integrable systems. We have not achieved this so far but work in this direction is now in progress. However, the property has the quality of being constructive. That is to say, it not only serves as a test of integrability but also provides us with the integrals of the dynamical system.

We must say, however, that our viewpoint differs from that of Sahadevan and Lakshmanan (1986) since we only consider the contact symmetry group of the Lagrangian and not the whole group of invariance of the equations of motion. The reason for this lies in the fact that it is the former, and not the latter, which really matters for integrability and from which the Noether invariants are obtained directly.

Let us consider a two-dimensional Lagrangian system given by

$$L=\tfrac{1}{2}(\dot{x}_1^2+\dot{x}_2^2)-V(x_1,x_2). \tag{1}$$

In order for this dynamical system to be (quasi-)invariant under the infinitesimal contact transformations

$$x_1'=x_1+\varepsilon\psi_1(x_i,\dot{x}_i,t) \tag{2a}$$

$$x_2'=x_2+\varepsilon\psi_2(x_i,\dot{x}_i,t) \qquad i=1,2 \tag{2b}$$

$$t'=t+\varepsilon\varphi(x_i,\dot{x}_i,t) \tag{2c}$$

then the following equation, which also defines Λ, must hold (Lutzky 1979):

$$S(L)+L\dot{\varphi}\equiv \mathrm{d}\Lambda/\mathrm{d}t \tag{3}$$

where $S(\;)=\varphi\,\partial/\partial t+\psi_i\,\partial/\partial x_i+(\dot{\psi}_i-\dot{x}_i\dot{\varphi})\,\partial/\partial\dot{x}_i$ is the extended infinitesimal generator for this contact symmetry group and $\Lambda\equiv\Lambda(x_i,\dot{x}_i,t)$ is a function to be determined from (3). Once we have solved (3) we obtain the constants of motion as

$$I\equiv(\psi_i-\dot{x}_i\varphi)\,\partial L/\partial\dot{x}_i+\varphi L-\Lambda \tag{4}$$

where we sum over dummy indexes. In general, this expression will depend on arbitrary constants; let us call them α, β, $\gamma,\ldots$ in the following way:

$$I\equiv\alpha I_1(x_i,\dot{x}_i,t)+\beta I_2(x_i,\dot{x}_i,t)+\gamma I_3(x_i,\dot{x}_i,t)+\ldots.$$

As the parameters α, β, γ are free, if I has to be an integral of the motion, it is necessary that any of the I separately be also a constant of motion. So, in this way, we obtain not one, but as many integrals of motion as arbitrary constants on which I depends.

We shall then look for a solution of equation (3). While it is possible to obtain the general solution of these equations for a quadratic Lagrangian (Cerveró and Villarroel 1984), it would be too optimistic to hope to do so for a higher-degree potential. Thus, we have to find a useful ansatz. Let us suppose that φ and ψ_i are given by

$$\varphi=a_0+a_1\dot{x}_1+a_2\dot{x}_2 \tag{5a}$$

$$\psi_1=b_0+b_1\dot{x}_1+b_2\dot{x}_2 \tag{5b}$$

$$\psi_2=c_0+c_1\dot{x}_1+c_2\dot{x}_2 \tag{5c}$$

where a_i, b_i and c_i $(i=0,1,2)$ are general functions of x_1, x_2 and t, but not of $\dot{x}_1$, $\dot{x}_2$. Thus, when inserting (5) into (3) every coefficient of $\dot{x}^m\dot{y}^n$, m, $n=1, 2, 3, 4$, must be set equal to zero, which gives rise to the following system of PDE:

$$\frac{\partial a_1}{\partial x_1}=\frac{\partial a_2}{\partial x_2}=0 \qquad \frac{\partial a_1}{\partial x_2}+\frac{\partial a_2}{\partial x_1}=0 \tag{6a}$$

$$\frac{\partial b_1}{\partial x_1}-\frac{1}{2}\left(\frac{\partial a_0}{\partial x_1}+\frac{\partial a_1}{\partial t}\right)=\frac{\partial c_2}{\partial x_2}-\frac{1}{2}\left(\frac{\partial a_0}{\partial x_2}+\frac{\partial a_2}{\partial t}\right)=0 \tag{6b}$$

$$\frac{\partial b_1}{\partial x_2}+\frac{\partial b_2}{\partial x_1}+\frac{\partial c_1}{\partial x_1}-\frac{1}{2}\left(\frac{\partial a_0}{\partial x_2}+\frac{\partial a_2}{\partial t}\right)=0 \tag{6c}$$

$$\frac{\partial c_1}{\partial x_2}+\frac{\partial c_2}{\partial x_1}+\frac{\partial b_2}{\partial x_2}-\frac{1}{2}\left(\frac{\partial a_0}{\partial x_1}+\frac{\partial a_1}{\partial t}\right)=0 \tag{6d}$$

$$\frac{\partial b_0}{\partial x_2}+\frac{\partial b_2}{\partial t}+\frac{\partial c_0}{\partial x_1}+\frac{\partial c_1}{\partial t}=0 \tag{6e}$$

$$\frac{\partial b_0}{\partial x_1}+\frac{\partial b_1}{\partial t}-\frac{1}{2}\left[\frac{\partial a_0}{\partial t}+a_1\left(\frac{\partial V}{\partial x_1}\right)+a_2\left(\frac{\partial V}{\partial x_2}\right)\right]=0 \tag{6f}$$

$$\frac{\partial c_0}{\partial x_2}+\frac{\partial c_2}{\partial t}-\frac{1}{2}\left[\frac{\partial a_0}{\partial t}+a_1\left(\frac{\partial V}{\partial x_1}\right)+a_2\left(\frac{\partial V}{\partial x_2}\right)\right]=0 \tag{6g}$$

$$\frac{\partial b_0}{\partial t}+2b_1\frac{\partial V}{\partial x_1}+(c_1+b_2)\frac{\partial V}{\partial x_2}+\left(\frac{\partial a_0}{\partial x_1}+\frac{\partial a_1}{\partial t}\right)V(x_1x_2)=\frac{\partial \Lambda}{\partial x_1} \tag{6h}$$

$$\frac{\partial c_0}{\partial t}+(c_1+b_2)\frac{\partial V}{\partial x_1}+2c_2\frac{\partial V}{\partial x_2}+\left(\frac{\partial a_0}{\partial x_2}+\frac{\partial a_2}{\partial t}\right)V(x_1x_2)=\frac{\partial \Lambda}{\partial x_2} \tag{6j}$$

$$b_0\frac{\partial V}{\partial x_1}+c_0\frac{\partial V}{\partial x_2}+\left(\frac{\partial a_0}{\partial t}+a_1\frac{\partial V}{\partial x_1}+a_2\frac{\partial V}{\partial x_2}\right)=\frac{\partial \Lambda}{\partial t}+\left(\frac{\partial \Lambda}{\partial x_1}\right)\left(\frac{\partial V}{\partial x_1}\right)+\left(\frac{\partial \Lambda}{\partial x_2}\right)\left(\frac{\partial V}{\partial x_2}\right). \tag{6k}$$

The function Λ has to be determined by demanding compatibility of (6h)-(6k).

In the following section we will consider separately the almost trivial case of separable potentials (for which the solution is known) to see whether the method works, and will then turn our attention to the more interesting case of non-linear non-separable potentials.

2. The separable case

Let us consider the Lagrangian given by

$$L=\tfrac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2)+P_1(x_1)+P_2(x_2) \tag{7}$$

where $P_1(x_1)$ and $P_2(x_2)$ are arbitrary polynomials of their arguments. Although this system is clearly integrable, it is interesting to know what is the subjacent symmetry if it exists. Inserting $V(x_1x_2)=P_1(x_1)+P_2(x_2)$ into the general system (6) we obtain after some lengthy work the general solution in the form

$$\varphi = a_0 \tag{8a}$$

$$\psi_1 = (\alpha + \tfrac{1}{2}a_0)\dot{x}_1 + b_2\dot{x}_2 + \beta \tag{8b}$$

$$\psi_2 = -b_2\dot{x}_1 + (\delta + \tfrac{1}{2}a_0)\dot{x}_2 + \gamma \tag{8c}$$

$$\Lambda = (2\alpha + a_0)P_1(x_1) + (2\delta + a_0)P_2(x_2) + \beta\dot{x}_1 + \gamma\dot{x}_2 \tag{8d}$$

where a_0 and b_2 are arbitrary functions of (x_1, x_2, t) and α, β, γ and δ are integration constants. Two constants of motion are found as†

$$I_1 = \tfrac{1}{2}\dot{x}_1^2 - P_1(x_1) \tag{9a}$$

$$I_2 = \tfrac{1}{2}\dot{x}_2^2 - P_2(x_2). \tag{9b}$$

As we have just seen, the group we have obtained depends on arbitrary functions a_0 and b_2 so it is an infinite-parameter group. As we shall see later on, this fact also holds for all cases we will consider. It is not clear to us if such a property can be a general one for integrable systems but we conjecture that this could be the case. If such were always true we would be able to assign this property as a fundamental feature of integrable dynamical systems.

3. The non-separable case

Let us now turn to the obviously more interesting non-separable case. Consider now in (1) a general form of the potential such as

$$V(x_1, x_2) = Ax_1^5 + Bx_1x_2^4 + Cx_1^3 + Dx_1x_2^2 + Ex_1 + Fx_1^3x_2^2 \tag{10}$$

where A, B, C, D, E and F are, for the time being, arbitrary constants. Clearly the Lagrangian (1) for the potential (10) will not be integrable for any value of such constants. On the other hand, the invariance conditions (6) are not in general compatible except for some particular cases. They are, however, compatible for precisely those cases when the system is integrable. Inserting (10) into the system (6) we find it to be fulfilled if and only if

$$A = F \qquad B = \tfrac{3}{16}F \qquad C = 2D. \tag{11}$$

The symmetry generators are given by

$$\varphi = a_0 \tag{12a}$$

$$\psi_1 = (\alpha + \tfrac{1}{2}a_0)\dot{x}_1 + b_2\dot{x}_2 \tag{12b}$$

$$\psi_2 = -b_2\dot{x}_1 + (\alpha + \beta x_1 + \tfrac{1}{2}a_0)\dot{x}_2 \tag{12c}$$

where a_0 and b_2 are again arbitrary functions of (x_1, x_2, t) while α and β are integration constants. The integral associated with this symmetry is

$$I = \dot{x}_1x_2\dot{x}_2 - x_1\dot{x}_2^2 + (\tfrac{1}{2}x_1^4x_2^2 + \tfrac{3}{8}x_1^2x_2^4 + \tfrac{1}{32}x_2^6)A + (\tfrac{1}{4}x_2^4 + x_1^2x_2^2 + \tfrac{1}{2}(E/D)x_2^2)D. \tag{13}$$

Another potential we have considered is

$$V(x_1, x_2) = -\tfrac{1}{6}(Ax_1^6 + Bx_2^6) - Cx_1^4x_2^2 - x_1^2x_2^4 \tag{14}$$

† Actually, the integral of motion (4) now becomes $I \equiv 2\alpha I_1 + 2\delta I_2$.

where we have set the last coefficient to one since such a rescaling does not modify integrability. Working out the invariance conditions we find that they are fulfilled in only two cases:

(i) $A = B = 2$, $C = 1$ (central potential)

(ii) $A = \frac{3}{40}$, $B = \frac{24}{5}$, $C = \frac{3}{10}$.

Case (i) is the trivial central force problem invariant under rotations. For case (ii) the symmetry generators are the same as for (12*a*, *b*, *c*). We can also find the following invariant:

$$I = -\dot{x}_1^2 x_2 + x_1 \dot{x}_1 \dot{x}_2 - (\tfrac{3}{40} x_1^6 x_2 + \tfrac{2}{5} x_1^2 x_2^5 + \tfrac{2}{5} x_1^4 x_2^3). \tag{15}$$

Finally consider the potential given in (Ramani *et al* 1982)

$$V(x_1 x_2) = \sum_{j=0}^{[n/2]} \binom{n-j}{j} 2^{n-2j} x_1^{2j} x_2^{n-2j}.$$

Our technique also works for this case yielding the invariant

$$I = -x_2 \dot{x}_1^2 + x_1 \dot{x}_1 \dot{x}_2 + \sum_{j=0}^{[n/2]} 2^{n-2j+1} \binom{n-j}{j} x_1^{2j} x_2^{n-2j+1} \left(\frac{j}{n-2j+1} \right) \tag{16}$$

and the symmetry generators are

$$\varphi = a_0 \tag{17a}$$

$$\psi_1 = (\tfrac{1}{2} a_0 + \alpha + \beta x_2) \dot{x}_1 + b_2 \dot{x}_2 \tag{17b}$$

$$\psi_2 = -(\beta + b_2) \dot{x}_1 + (\alpha + \tfrac{1}{2} a_0) \dot{x}_2. \tag{17c}$$

We would like to show that our method also works for other kinds of potentials in more than two dimensions. As an example (Calogero 1969) consider

$$L = \tfrac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - g \left(\frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 - x_3)^2} + \frac{1}{(x_2 - x_3)^2} \right). \tag{18}$$

In this case, the compatibility of the system (6) indicates that it is sufficient to consider only point transformations. Therefore, restricting ourselves to these kind of transformations we obtain the following set of symmetry generators:

$$S_1 = \partial/\partial t \tag{19a}$$

$$S_2 = t\, \partial/\partial t + x_i\, \partial/\partial x_i \tag{19b}$$

$$S_3 = \tfrac{1}{2} t^2\, \partial/\partial t + t x_i\, \partial/\partial x_i \tag{19c}$$

$$S_4 = \partial/\partial x_1 + \partial/\partial x_2 + \partial/\partial x_3 \tag{19d}$$

$$S_5 = t(\partial/\partial x_1 + \partial/\partial x_2 + \partial/\partial x_3) \qquad i = 1, 2, 3. \tag{19e}$$

As expected, S_1, S_2 and S_3 close the conformal one-dimensional group SO(2, 1). The associated constants of motion are

$$I_1 = \tfrac{1}{2} \dot{x}_i \dot{x}^i + g \left(\frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 - x_3)^2} + \frac{1}{(x_2 - x_3)^2} \right) \tag{20a}$$

$$I_2 = t I_1 - \tfrac{1}{2} x_i \dot{x}^i \tag{20b}$$

$$I_3 = \tfrac{1}{2} t^2 I_1 - \tfrac{1}{2} t x_i \dot{x}^i + \tfrac{1}{4} (x_1^2 + x_2^2 + x_3^2) \tag{20c}$$

$$I_4 = \dot{x}_1 + \dot{x}_2 + \dot{x}_3 \tag{20d}$$

$$I_5 = x_1 + x_2 + x_3 - t I_4. \tag{20e}$$

which, although being explicitly time dependent, justify the integrability of the system. In fact, introducing the new coordinates

$$R = \tfrac{1}{3}(x_1 + x_2 + x_3) \tag{21a}$$

$$\xi = (1/\sqrt{6})(x_1 + x_2 - 2x_3) \tag{21b}$$

$$\eta = (1/\sqrt{2})(x_1 - x_2) \tag{21c}$$

and using one of the integrals to solve for time (say I_2) while another (say I_5) is used to solve for R, we arrive at the following set of first integrals:

$$H = \tfrac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2) + \tfrac{9}{2}g\left(\frac{\xi^2 + \eta^2}{\eta^2(3\xi^2 - \eta^2)}\right) \tag{22a}$$

$$B^2 = (\xi^2 + \eta^2)H - \tfrac{1}{2}(\dot{\xi}\xi + \eta\dot{\eta}) \tag{22b}$$

$$P = \dot{R} = \tfrac{1}{3}(\dot{x}_1 + \dot{x}_2 + \dot{x}_3) \tag{22c}$$

from which (Marchioro 1970) we would obtain the explicit solution of the equations of motion.

We think this method can in principle work for any non-linear system restricted to a finite number of degrees of freedom as well as for polynomial-type potentials. Indeed, it could also be extended to non-polynomial interactions but then the difficulty of solving (6) would be considerably increased.

4. Conclusions

The analysis above points out that the existence of symmetry transformations is an inherent property of integrable systems. Such a symmetry must in principle be of contact nature although in some simple cases the transformation could be only point-like. The method is also constructive since once we have identified the symmetries of the dynamical system, we can integrate it readily by using Noether's theorem. All these reasons point in the direction that contact invariance properties seem to be an unavoidable and useful ingredient in the study of integrability in non-linear dynamical systems.

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